

Cohomology of diagrams of algebras

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Abstract

We consider cohomology of diagrams of algebras by Beck's approach, using comonads. We then apply this theory to computing the cohomology of Ψ -rings. Our main result is that there is a spectral sequence connecting the cohomology of the diagram of an algebra to the cohomology of the underlying algebra.

1 Introduction

Algebraic objects such as groups, Lie algebras, associative algebras and commutative algebras have cohomology theories defined in its own way. For example, group cohomology was defined by Eilenberg-MacLane [10], Lie algebra cohomology was defined by Chevalley-Eilenberg [8], associative algebra cohomology was defined by Hochschild [13], and commutative algebra cohomology was defined by André [1], Quillen [16], and Barr [3]. In the 1960's it was found that all of these can be defined in one scheme. Here, we are going to use Barr-Beck's approach, which is based on comonads. Let $T : \mathbf{Sets} \rightarrow \mathbf{Sets}$ be a monad. Then one can consider the category of T -algebras, $\mathbf{Alg}(T)$, over the monad T . Let A be a T -algebra and M be an A -module, which by definition is an abelian group object in $\mathbf{Alg}(T)/A$. One defines the cohomology of A with coefficients in M , $H_{\mathbb{G}}^*(A, M)$, as follows. There exists a pair of adjoint functors:

$$\mathbf{Sets} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \mathbf{Alg}(T)$$

where U is the forgetful functor. This adjoint pair of functors yields a comonad $\mathbb{G} = FU : \mathbf{Alg}(T) \rightarrow \mathbf{Alg}(T)$. One can take the comonad resolution $\mathbb{G}_*(A)$. One can then apply the functor $Der(-, M)$. One can then define a cochain complex by taking the alternating sum of the induced maps, and then one can take the cohomology. This situation is very general, one would like to apply this in the case of λ -rings and Ψ -rings.

A ψ -ring is a commutative ring R with identity 1, together with a series of homomorphisms $\psi^n : R \rightarrow R$, $n \geq 1$ such that $\forall x \in R$ and integers $n, m \geq 1$ one has $\psi^1(x) = x$ and $\psi^n(\psi^m(x)) = \psi^{nm}(x)$. So a ψ -ring can be thought of as a diagram of an algebra with the monoid of the natural numbers acting on R .

Recently, several authors have defined cohomology of diagrams of algebras. For instance, the cohomology of diagrams of groups has been considered by Cegarra [7]. Cohomology of diagrams of Π -algebras has been considered by Blanc, Johnson, and Turner [6]. Cohomology of diagrams of associative algebras was considered by Gerstenhaber and Schack [12]. Cohomology of diagrams of Lie algebras was considered by Gerstenhaber and Schack [11]. All of these cohomologies are defined in its own way and it is not clear how to make them as a particular case of one general theory.

The first aim is to use the Bar-Beck approach to unify the cohomology of diagrams of algebras. Secondly, we would like to relate the diagram cohomology to the cohomology of the algebra using a local to global spectral sequence. Thirdly, we would like to apply the theory to the category of commutative rings to study the cohomology of ψ -rings.

Our approach to defining the cohomology of diagrams of algebras can be described as follows. First, fix a small category I . A diagram of algebras is a functor $I \rightarrow \mathfrak{Alg}(T)$, where T is above a monad on sets. For appropriate T , one gets diagram of groups, diagram of Lie algebras, commutative rings, etc.

One considers also the category I_0 , which has the same objects as I , but only the identity morphisms. The obvious inclusion $I_0 \subset I$ yields the functor $\mathfrak{Sets}^I \rightarrow \mathfrak{Sets}^{I_0}$ which has left adjoint (the left Kan extension). We also have the pair of adjoint functors $\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{Sets}^I$ which comes from the adjoint pair $\mathfrak{Alg}(T) \rightleftarrows \mathfrak{Sets}$. By gluing these diagrams, one gets another adjoint pair

$$\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{Sets}^{I_0}$$

We will prove that $\mathfrak{Alg}(T)^I$ is monadic in \mathfrak{Sets}^{I_0} and the right cohomology theory of diagrams of algebras is one which is associated to the corresponding comonad. These cohomology theories are denoted by $H_I^*(A, M)$. We will prove that the comonad cohomology is isomorphic to the ones considered in [7] and [6] by choosing appropriate T .

The main technical element for studying $H_I^*(A, M)$ is the local to global spectral sequence which can be described as follows. Let $\mathbf{A} : I \rightarrow \mathfrak{Alg}(T)$ be a diagram of T -algebras and M is an \mathbf{A} -module. In particular, for each $i \in I$ one has $M(i)$ an $A(i)$ -module and for any arrow $\alpha : i \rightarrow j$ one can consider $M(j)$ as an $A(i)$ -module where $A(i)$ acts on $M(j)$ via the algebra homomorphism $A(\alpha) : A(i) \rightarrow A(j)$. This allows us to consider the cohomology $\mathcal{H}_{\mathbb{G}}^n(A, M)(\alpha)$ defined to be:

$$H_{\mathbb{G}}^n(A(i), \alpha^* M(j))$$

In this way we get a natural system [4] on I .

Our main result claims that there exists a spectral sequence:

$$E_2^{pq} = H_{BW}^p(I, \mathcal{H}_{\mathbb{G}}^q(A, M)) \Rightarrow H_I^*(A, M).$$

where on the left hand side one uses the Baues-Wirsching cohomology of a small category with coefficients in a natural system. This spectral sequence is new in

almost all of cases and gives computational tools for diagram cohomology; even for diagrams of groups, diagrams of associative algebras, diagrams of Π -algebras, etc.

2 Prerequisites

2.1 Baues Wirsching Cohomology

For a small category \mathcal{I} . The *category of factorizations* in \mathcal{I} , denoted by \mathcal{FI} , is the category with objects the morphisms f, g, \dots in \mathcal{I} , and morphisms $f \rightarrow g$ are pairs (α, β) of morphisms in \mathcal{I} such that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B' \\ f \uparrow & & \uparrow g \\ A & \xleftarrow{\beta} & A' \end{array}$$

Composition in \mathcal{FI} is given by $(\alpha', \beta')(\alpha, \beta) = (\alpha'\alpha, \beta\beta')$. A *natural system* of abelian groups on \mathcal{I} is a functor from the category of factorizations to the category of abelian groups:

$$D : \mathcal{FI} \rightarrow \mathfrak{Ab}$$

There are natural functors:

$$\begin{array}{ccccc} \mathcal{FI} & \longrightarrow & \mathcal{I}^{op} \times \mathcal{I} & \longrightarrow & \mathcal{I} \\ & & \downarrow & & \downarrow \\ & & \mathcal{I}^{op} & \longrightarrow & \mathbf{1} \end{array}$$

which allows one to consider any functor or bifunctor on \mathcal{I} as a natural system on \mathcal{I} . Following Baues-Wirsching [4], one defines the cohomology $H_{BW}^*(\mathcal{I}, D)$ of \mathcal{I} with coefficients in the natural system D as the cohomology of the cochain complex $C_{BW}^*(\mathcal{I}, D)$ given by:

$$C_{BW}^n(\mathcal{I}, D) = \prod_{\alpha_1 \dots \alpha_n : i_n \rightarrow \dots \rightarrow i_0} D(\alpha_1 \dots \alpha_n)$$

and the coboundary map

$$d : C_{BW}^n(\mathcal{I}, D) \rightarrow C_{BW}^{n+1}(\mathcal{I}, D)$$

is given by:

$$\begin{aligned} (df)(\alpha_1 \dots \alpha_{n+1}) &= (\alpha_1)_* f(\alpha_2, \dots, \alpha_{n+1}) \\ &+ \sum_{j=1}^n (-1)^j f(\alpha_1, \dots, \alpha_j \alpha_{j+1}, \dots, \alpha_{n+1}) \\ &+ (-1)^{n+1} (\alpha_{n+1})^* f(\alpha_1, \dots, \alpha_n) \end{aligned}$$

We will need the following well-known lemma later:

Lemma 2.1 Assume $i_0 \in \mathcal{I}$ is an initial object, and $F : \mathcal{I} \rightarrow \mathbf{Ab}$ a functor. Then:

$$H_{BW}^n(\mathcal{I}, F) = \begin{cases} F(i_0) & n = 0 \\ 0 & n > 0 \end{cases}$$

2.2 Base change

Let \mathfrak{C} be a category, and X an object in \mathfrak{C} . An X -module in \mathfrak{C} is an abelian group object in the category \mathfrak{C}/X .

$$X - \text{mod} := \mathbf{Ab}(\mathfrak{C}/X)$$

Theorem 2.2 Let $f : X \rightarrow Y$ be a morphism in \mathfrak{C} , then there exists a base-change functor $f^* : Y - \text{mod} \rightarrow X - \text{mod}$ via pullbacks.

Proof. The functor we are going to use is $f^* : \mathfrak{C}/Y \rightarrow \mathfrak{C}/X$ given by pullbacks:

$$\begin{array}{ccc} f^*(M) & \longrightarrow & M \\ \downarrow & & \downarrow p \\ X & \xrightarrow{f} & Y \end{array}$$

If $M \in Y - \text{mod}$ then $f^*(M)$ has a canonical X -module structure. In set-theoretic notation:

$$f^*(M) = \{(x, m) | x \in X, m \in M, f(x) = p(m)\}$$

$$f^*(M) \times_X f^*(M) = \{(x, m, m') | x \in X, m, m' \in M, f(x) = p(m) = p(m')\}$$

$$f^*(M) \times_X f^*(M) \simeq f^*(M \times_Y M)$$

Consider the following commuting diagram.

$$\begin{array}{ccccc} f^*(M \times_Y M) & \longrightarrow & M \times_Y M & & \\ \downarrow & \searrow \exists! & \downarrow & \searrow \text{mult} & \\ X & \xrightarrow{f} & Y & & \\ & & \downarrow & \searrow & \\ & & f^*(M) & \longrightarrow & M \\ & & \downarrow & & \downarrow \\ & & X & \xrightarrow{f} & Y \end{array}$$

The unique morphism $f^*(\text{mult}) : f^*(M \times_Y M) \rightarrow f^*(M)$ exists by the universal property of pullbacks. The isomorphism $f^*(M) \times_X f^*(M) \simeq f^*(M \times_Y M)$ and this unique morphism yield multiplication:

$$f^*(\text{mult}) : f^*(M) \times_X f^*(M) \rightarrow f^*(M)$$

which gives an abelian group object structure on $f^*(M)$. □

2.3 Derivations

For $M \in X\text{-mod}$, one defines a *derivation* from X to M to be a morphism $d : X \rightarrow M$ which is a section of the canonical morphism $M \rightarrow X$. Let $\text{Der}(X, M)$ denote the set of derivations $d : X \rightarrow M$. We will require the following useful theorem later.

Theorem 2.3 *If $X = \coprod_{\alpha \in I} X_\alpha$ and $M \in X\text{-mod}$, then $\text{Der}(X, M) \cong \prod_{\alpha \in I} \text{Der}(X_\alpha, M_\alpha)$, where M_α is an X_α -module by the base-change functor from the morphism $i_\alpha : X_\alpha \rightarrow X$.*

Proof. From the definition of the coproduct one has a morphism $i_\alpha : X_\alpha \rightarrow X$. Using this one gets $M_\alpha \in X_\alpha\text{-mod}$ via the following pullback diagram.

$$\begin{array}{ccc} M & \xleftarrow{j_\alpha} & M_\alpha \\ p \downarrow & & \downarrow p_\alpha \\ X & \xleftarrow{i_\alpha} & X_\alpha \end{array}$$

Let f be a section of p , this means that $pf = id_X$. Consider the following diagram:

$$\begin{array}{ccccc} & & & & X_\alpha \\ & & & \nearrow f i_\alpha & \\ & & & \nearrow f_\alpha & \\ M & \xleftarrow{j_\alpha} & M_\alpha & & \\ \uparrow f & & \downarrow p_\alpha & \nearrow id_{X_\alpha} & \\ p \downarrow & & & & \\ X & \xleftarrow{i_\alpha} & X_\alpha & & \end{array}$$

The diagram commutes since $pf i_\alpha = id_X i_\alpha = i_\alpha id_{X_\alpha}$. By the universal property of pullbacks $p_\alpha f_\alpha = id_{X_\alpha}$. So if f is a section of p then f_α is a section of p_α .

Conversely, let f_α be a section of p_α , this means that $p_\alpha f_\alpha = id_{X_\alpha}$. By the definition of the coproduct there exists a unique f such that the following diagram commutes:

$$\begin{array}{ccc} M & & \\ \uparrow f & \nearrow j_\alpha f_\alpha & \\ X & \xleftarrow{i_\alpha} & X_\alpha \end{array}$$

This means that $f i_\alpha = j_\alpha f_\alpha$. Composing with p on the left gives us that $pf i_\alpha = p j_\alpha f_\alpha = i_\alpha p_\alpha f_\alpha = i_\alpha id_{X_\alpha} = i_\alpha$. Thus the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{id_X} & X \\ \uparrow i_\alpha & \nearrow pf & \nearrow i_\alpha \\ X_\alpha & & \end{array}$$

The universal property of the coproduct says that $pf = id_X$. Hence f is a section of p . \square

2.4 Coequalizers

Let $f, g : a \rightarrow b$ be a pair in \mathfrak{C} , a *coequalizer* of $\langle f, g \rangle$ is an arrow $u : b \rightarrow c$ such that:

1. $uf = ug$
2. if $h : b \rightarrow d$ with $hf = hg$, then there exists a unique $h' : c \rightarrow d$ such that $h = h'u$:

$$\begin{array}{ccccc} a & \xrightleftharpoons[g]{f} & b & \xrightarrow{u} & c \\ & & \searrow h & & \vdots h' \\ & & & & d \end{array}$$

u is called a *split coequalizer* of f and g if u is a coequalizer of f and g , and in addition there exists $s : c \rightarrow b$ and $t : b \rightarrow a$ such that $us = 1$ $ft = 1$, and $gt = su$:

$$\begin{array}{ccccc} a & \xrightleftharpoons[g]{f} & b & \xrightarrow{u} & c \\ \curvearrowright t & & \curvearrowleft s & & \end{array}$$

2.5 Comonad Cohomology

Let \mathfrak{C} be a category, and $\mathbb{G} = (G : \mathfrak{C} \rightarrow \mathfrak{C}, \varepsilon : G \rightarrow Id, \delta : G \rightarrow G^2)$ be a comonad on \mathfrak{C} . For an object X in \mathfrak{C} , the comonad gives rise to a functorial augmented simplicial object over X , which we denote by $\mathbb{G}_*(X) \rightarrow X$. The object of $\mathbb{G}_*(X)$ in degree n is $G^{n+1}(X)$, and the maps are $\varphi_i = G^i \varepsilon G^{n-i} X : G^{n+1}(X) \rightarrow G^n(X)$, and $\sigma_i = G^i \delta G^{n-i} X : G^{n+1}(X) \rightarrow G^{n+2}(X)$ for $0 \leq i \leq n$. X itself is in dimension -1 and the augmenting map is just ε .

$$\cdots \begin{array}{c} \longrightarrow \\ \vdots \\ \longrightarrow \end{array} G^n X \begin{array}{c} \longrightarrow \\ \vdots \\ \longrightarrow \end{array} G^{n-1} X \begin{array}{c} \longrightarrow \\ \vdots \\ \longrightarrow \end{array} \cdots \xrightleftharpoons[\varepsilon G]{G \varepsilon} GX \xrightarrow{\varepsilon} X$$

For any $M \in X - mod := Ab(\mathfrak{C}/X)$, one can apply the functor $Der(-, M)$ and take the alternating sum of the induced homomorphisms to get the cochain complex whose cohomology is defined to be $H_{\mathbb{G}}^*(X, M)$.

A morphism $f : X \rightarrow Y$ in \mathfrak{C} is called a \mathbb{G} -*epimorphism* if the map $Hom_{\mathfrak{C}}(G(Z), X) \rightarrow Hom_{\mathfrak{C}}(G(Z), Y)$ is surjective for all Z . We require the following useful lemma:

Lemma 2.4 $GX \xrightarrow{\varepsilon_X} X$ is a \mathbb{G} -epimorphism.

Proof:. For any map $h : GZ \rightarrow X$, we wish to find a map $f : GZ \rightarrow GX$ such that $f\varepsilon_X = h$. We define f via the following commutative diagram:

$$\begin{array}{ccccc} & & G(GZ) & \xrightarrow{G(h)} & G(X) & \xrightarrow{\varepsilon_X} & X \\ & \delta(Z) \uparrow & & \nearrow f & & & \\ & GZ & & & & & \end{array}$$

Now we need to check that $f\varepsilon_X = h$. By the naturality of ε , the following diagram commutes:

$$\begin{array}{ccccc} & & GX & \xrightarrow{\varepsilon_X} & X \\ & \nearrow f & \uparrow G(h) & & \uparrow h \\ & G(GZ) & \xrightarrow{\varepsilon_{GZ}} & GZ & \\ & \delta(Z) \uparrow & \nearrow id_{GZ} & & \\ & GZ & & & \end{array}$$

So ε_X is a \mathbb{G} -epimorphism. □

An object P of \mathfrak{C} is called \mathbb{G} -projective if for any \mathbb{G} -epimorphism $f : X \rightarrow Y$, the map $Hom_{\mathfrak{C}}(P, X) \rightarrow Hom_{\mathfrak{C}}(P, Y)$ is surjective. Later we will require the following lemmas:

Example 2.5 For all Z , $G(Z)$ is \mathbb{G} -projective.

Lemma 2.6 The coproduct of \mathbb{G} -projective objects is \mathbb{G} -projective.

Proof:. Let $P = \coprod_i P_i$ where P_i is \mathbb{G} -projective for all i . For a map $f : X \rightarrow Y$, one applies the functors $Hom_{\mathfrak{C}}(P, -)$ and $Hom_{\mathfrak{C}}(P_i, -)$ to get the maps $f_* : Hom_{\mathfrak{C}}(P, X) \rightarrow Hom_{\mathfrak{C}}(P, Y)$, $f_{i*} : Hom_{\mathfrak{C}}(P_i, X) \rightarrow Hom_{\mathfrak{C}}(P_i, Y)$. If f is a \mathbb{G} -epimorphism then f_{i*} is surjective for all i . Using the well-known lemma $Hom_{\mathfrak{C}}(\coprod_i P_i, Z) \cong \prod_i Hom_{\mathfrak{C}}(P_i, Z)$ one gets that if f is a \mathbb{G} -epimorphism then $f_* \cong \prod_i f_{i*}$ is surjective. Hence P is \mathbb{G} -projective if P_i is \mathbb{G} -projective for all i . □

Lemma 2.7 An object P is \mathbb{G} -projective if and only if P is a retract of an object of the form $G(Z)$.

Proof:. A retract of a surjective map is surjective, so it is sufficient to consider the case $P = G(Z)$, which is obvious from the definition of \mathbb{G} -epimorphism. □

Lemma 2.8 $H_{\mathbb{G}}^p(X, M) = 0$, for $p > 0$ provided X is \mathbb{G} -projective.

Proof:. From the previous lemma, it is sufficient to check this in the case where $X = G(Z)$. In this case it is possible to construct a contracting homotopy. There

are maps $s_n : G^{n+2} \rightarrow G^{n+3}$ for $n \geq -1$ such that $\epsilon s_{-1} = id$, $\varphi_{n+1} s_n = id$, $\varphi_0 s_0 = s_{-1} \epsilon$, and $\varphi_i s_n = s_{n-1} \varphi_i$ for all $0 \leq i \leq n$.

$$s_n = G^{n+1} \delta$$

It follows that $H_{\mathbb{G}}^p(G(Z), M) = 0$, for $p > 0$. \square

Lemma 2.9 $H_{\mathbb{G}}^0(X, M) = Der(X, M)$ for all X .

3 T -algebras

Start with an adjunction $\mathfrak{S}ets \rightarrow \mathfrak{C}$, and construct the monad $T : \mathfrak{S}ets \rightarrow \mathfrak{S}ets$ defined in \mathfrak{C} . Then one can consider the category of T -algebras, $\mathfrak{Alg}(T)$, over the monad T . There exists an adjoint pair of functors:

$$\mathfrak{S}ets \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \mathfrak{Alg}(T)$$

where U is the forgetful functor. This adjoint pair of functors yields a comonad $\mathbb{G} = FU : \mathfrak{Alg}(T) \rightarrow \mathfrak{Alg}(T)$. There exists a unique functor $K : \mathfrak{C} \rightarrow \mathfrak{Alg}(T)$. If K is an equivalence of categories, then one says that \mathbb{G} is *monadic*.

Theorem 3.1 (Beck's Theorem [14]) *The following are equivalent:*

1. *The comparison functor $K : \mathfrak{C} \rightarrow \mathfrak{Alg}(T)$ is an equivalence of categories.*
2. *If f, g is any parallel pair in \mathfrak{C} for which Uf, Ug has a split coequalizer, then \mathfrak{C} has a coequalizer for f, g , and U preserves and reflects coequalizers for these pairs.*

Fix a small category I , one also considers the category I_0 , which has the same objects as I , but only the identity morphisms. The obvious inclusion $I_0 \subset I$ yields the functor $\mathfrak{S}ets^I \rightarrow \mathfrak{S}ets^{I_0}$ which has left adjoint (the left Kan extension). Let $\mathcal{F} : I_0 \rightarrow \mathfrak{S}ets$ be a functor, then the left Kan extension $Lan(\mathcal{F}) : I \rightarrow \mathfrak{S}ets$ is given by $Lan(\mathcal{F})(i) = \coprod_{x \rightarrow i} \mathcal{F}(x)$. We also have the pair of adjoint functors $\mathfrak{Alg}(T)^I \rightleftarrows \mathfrak{S}ets^I$ which comes from the adjoint pair $\mathfrak{Alg}(T) \rightleftarrows \mathfrak{S}ets$. By gluing these diagrams, one gets another adjoint pair

$$\mathfrak{Alg}(T)^I \begin{array}{c} \xrightarrow{U_I} \\ \xleftarrow{F_I} \end{array} \mathfrak{S}ets^{I_0}$$

This adjoint pair of functors yields a comonad $\mathbb{G}_I = F_I U_I : \mathfrak{Alg}(T)^I \rightarrow \mathfrak{Alg}(T)^I$. If $A : I \rightarrow \mathfrak{Alg}(T)$, then

$$\mathbb{G}_I(A)(i) = \coprod_{x \rightarrow i} \mathbb{G}(x)$$

Lemma 3.2 *If \mathbb{G} is monadic, then \mathbb{G}_I is monadic.*

Proof:. Assume \mathbb{G} is monadic and consider a parallel pair f, g in \mathfrak{C}^I :

$$F \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} T$$

If there is a split coequalizer in X as follows:

$$UF \begin{array}{c} \xrightarrow{Uf} \\ \xrightarrow{Ug} \end{array} UT \xrightarrow{q} W$$

then by theorem 3.1 for each $i \in I$, one has that $q(i)$ is a coequalizer of the following:

$$UF(i) \begin{array}{c} \xrightarrow{Uf(i)} \\ \xrightarrow{Ug(i)} \end{array} UT(i)$$

Hence by theorem 3.1 there exists $Z(i)$, $h(i)$ in \mathfrak{C} with :

$$F(i) \begin{array}{c} \xrightarrow{f(i)} \\ \xrightarrow{g(i)} \end{array} T(i) \xrightarrow{h(i)} Z(i)$$

such that $UZ(i) = W(i)$ and $Uh(i) = q(i)$. In fact Z is a functor $I \rightarrow \mathfrak{C}$. For $\alpha : i \rightarrow j$ one considers the following commuting diagram for $(\alpha : i \rightarrow j) \in I$:

$$\begin{array}{ccccc} F(i) & \begin{array}{c} \xrightarrow{f(i)} \\ \xrightarrow{g(i)} \end{array} & T(i) & \xrightarrow{h(i)} & Z(i) \\ F(\alpha) \downarrow & & \downarrow T(\alpha) & & \downarrow \exists! \\ F(j) & \begin{array}{c} \xrightarrow{f(j)} \\ \xrightarrow{g(j)} \end{array} & T(j) & \xrightarrow{h(j)} & Z(j) \end{array}$$

Since the coequalizer is universal, this means that there exists a unique map $Z(\alpha) : Z(i) \rightarrow Z(j)$ which commutes with the above diagram. One can check that Z is indeed a functor. \square

Lemma 3.3 *For all objects Z in \mathfrak{C} , and for $A \in G(Z) - \text{mod}$, one has*

$$\text{Der}(G(Z), A) = \{s \in \mathfrak{S}\mathfrak{c}\mathfrak{t}\mathfrak{s}(UZ, UA) | U(\pi)s = j_{UZ} : U(Z) \rightarrow UFU(Z)\}$$

where π is the canonical morphism $\pi : A \rightarrow G(Z)$

Proof:. From the definition of F, U being an adjoint pair, one gets that:

$$\text{Hom}_{\mathfrak{A}\mathfrak{lg}(\mathbb{T})}(F(X), Y) = \text{Hom}_{\mathfrak{S}\mathfrak{c}\mathfrak{t}\mathfrak{s}}(X, U(Y))$$

Setting $X = U(Z)$ and $Y = FU(Z)$ we get the following:

$$\text{Hom}_{\mathfrak{A}\mathfrak{lg}(\mathbb{T})}(FU(Z), FU(Z)) = \text{Hom}_{\mathfrak{S}\mathfrak{c}\mathfrak{t}\mathfrak{s}}(U(Z), UFU(Z))$$

From this it can be shown that

$$\text{Der}(G(Z), A) = \{s \in \mathfrak{S}\mathfrak{c}\mathfrak{t}\mathfrak{s}(UZ, UA) | U(\pi)s = j_{UZ} : U(Z) \rightarrow UFU(Z)\}$$

\square

4 Cohomology of diagrams of algebras

In this section, let \mathfrak{C} denote the category of sets, and I denote a small category. \mathfrak{C} is a category with limits. We require the following useful theorem.

Theorem 4.1 *Let $A : I \rightarrow \mathfrak{C}$ be a functor, and $M \in A\text{-mod} := \text{Ab}(\mathfrak{C}^I/A)$ where \mathfrak{C}^I is the category of functors. If $(\alpha : i \rightarrow j) \in I$, then $M(j) \in A(j)\text{-mod}$ and*

$$\mathfrak{Der}(A, M)(\alpha) = \text{Der}(A(i), \alpha^* M(j))$$

defines a natural system on I .

Proof:. Start by fixing A and M , then let $D(\alpha)$ denote $\mathfrak{Der}(A, M)(\alpha)$. Let $\gamma, \alpha, \beta \in I$ such that:

$$i' \xrightarrow{\gamma} i \xrightarrow{\alpha} j \xrightarrow{\beta} j'$$

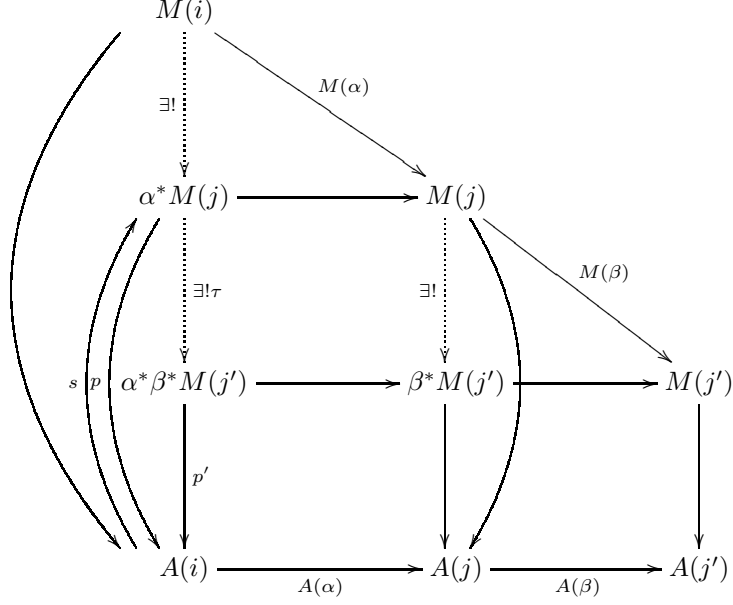
We are going to show that:

$$D(\alpha\gamma) \xleftarrow{\gamma^*} D(\alpha) \xrightarrow{\beta^*} D(\beta\alpha)$$

Let $s \in D(\alpha)$, then the following diagram commutes with $ps = \text{id}_{A(i)}$, and $\alpha^* M(j)$ is a pullback; $\alpha^* M(j) \in A(i)\text{-mod}$.

$$\begin{array}{ccc} \alpha^* M(j) & \xrightarrow{\quad} & M(j) \\ \uparrow s & & \downarrow p \\ A(i) & \xrightarrow{A(\alpha)} & A(j) \end{array}$$

Consider the following commuting diagram:



Let $s' : A(i) \rightarrow \alpha^* \beta^* M(j')$ be the map $s' = \tau s$. If we let $s \in D(\alpha)$, this means that $ps = id_{A(i)}$. Hence

$$p'\tau s = ps = id_{A(i)}$$

Hence $s' \in Der(A(i), \alpha^* \beta^* M(j')) = Der(A(i), (\beta\alpha)^* M(j'))$.

Consider the following commutative diagram, with s a section of p .

$$\begin{array}{ccccc}
 (\alpha\gamma)^* M(j) & \longrightarrow & \alpha^* M(j) & \longrightarrow & M(j) \\
 \downarrow p' & & \uparrow s & \downarrow p & \downarrow \\
 A(i') & \xrightarrow{A(\gamma)} & A(i) & \xrightarrow{A(\alpha)} & A(j)
 \end{array}$$

There exists a unique $s' : A(i') \rightarrow (\alpha\gamma)^* M(j)$ which is a section of p' which would make the above diagram still commute. Therefor $s' \in Der(A(i'), (\alpha\gamma)^* M(j))$.

□

Let \mathbb{G} be a comonad in \mathfrak{C} . Let $A : I \rightarrow \mathfrak{C}$ be a functor, and $M \in A\text{-mod}$. Then we can construct the following bicomplex denoted by $C^{*,*}(I, A, M)$:

$$C^{p,q}(I, A, M) = \prod_{i_0 \rightarrow \dots \rightarrow i_p} Der(G^{q+1}(A(i)), M(k))$$

The map $C^{p,q}(I, A, M) \rightarrow C^{p+1,q}(I, A, M)$ is the map in the Baues-Wirsching cochain complex, and the map $C^{p,q}(I, A, M) \rightarrow C^{p,q+1}(I, A, M)$ is the coproduct of maps in the comonad cochain complex.

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & \\
& \uparrow & & \uparrow & & \uparrow & \\
\Pi_i \operatorname{Der}(G^3(A(i)), M(i)) & \longrightarrow & \Pi_{i \rightarrow j} \operatorname{Der}(G^3(A(i)), M(j)) & \longrightarrow & \Pi_{i \rightarrow j \rightarrow k} \operatorname{Der}(G^3(A(i)), M(k)) & \longrightarrow & \dots \\
& \uparrow & & \uparrow & & \uparrow & \\
\Pi_i \operatorname{Der}(G^2(A(i)), M(i)) & \longrightarrow & \Pi_{i \rightarrow j} \operatorname{Der}(G^2(A(i)), M(j)) & \longrightarrow & \Pi_{i \rightarrow j \rightarrow k} \operatorname{Der}(G^2(A(i)), M(k)) & \longrightarrow & \dots \\
& \uparrow & & \uparrow & & \uparrow & \\
\Pi_i \operatorname{Der}(G(A(i)), M(i)) & \longrightarrow & \Pi_{i \rightarrow j} \operatorname{Der}(G(A(i)), M(j)) & \longrightarrow & \Pi_{i \rightarrow j \rightarrow k} \operatorname{Der}(G(A(i)), M(k)) & \longrightarrow & \dots
\end{array}$$

We let $H^*(I, A, M)$ denote the cohomology of the total complex of $C^{*,*}(I, A, M)$.

We will need the following useful lemmas:

Lemma 4.2 *If A is \mathbb{G}_I -projective, then $A(i)$ is \mathbb{G} -projective for all $i \in I$.*

Proof: Consider $A = G_I(Z) : I \rightarrow \mathfrak{C}$ where $G_I(A)(i) = \coprod_{x \rightarrow i} G(A(x))$. Since $G(A(x))$ is \mathbb{G} -projective, it follows that $\coprod_{x \rightarrow i} G(A(x))$ is \mathbb{G} -projective for all $i \in I$. \square

Lemma 4.3 *$H^0(I, A, M) = \mathfrak{D}\mathfrak{er}(A, M)$, furthermore, if A is \mathbb{G}_I -projective then $H^n(I, A, M) = 0$ for $n > 0$.*

Proof: It is sufficient to consider the case when $A = G_I(Z)$. When $A = G_I(Z)$, it is known that A is \mathbb{G}_I -projective. By theorems 4.2 and 2.8, one gets that the vertical columns in our bicomplex are exact except in dimension 0. There is a well known lemma for bicomplexes which tells us the cohomology of the total complex is isomorphic to the cohomology of the following chain complex:

$$\Pi_i \operatorname{Der}(A(i), M(i)) \longrightarrow \Pi_{i \rightarrow j} \operatorname{Der}(A(i), M(j)) \longrightarrow \dots$$

It is known that the cohomology of this cochain complex is just $H_{BW}^*(I, \mathfrak{D}\mathfrak{er}(A, M))$.

To prove the first statement it is enough to show that

$$0 \rightarrow \mathfrak{D}\mathfrak{er}(A, M) \rightarrow \prod_i \operatorname{Der}(A(i), M(i)) \rightarrow \prod_{\alpha: i \rightarrow j} \operatorname{Der}(A(i), M(j))$$

is exact. Let $\psi \in \prod_i \operatorname{Der}(A(i), M(i))$ and $(\alpha : i \rightarrow j) \in I$, then $d\psi(\alpha : i \rightarrow j) = \alpha_*\psi(i) - \alpha^*\psi(j)$. Therefore $d\psi(\alpha : i \rightarrow j) = 0$ if and only if $\alpha_*\psi(i) = \alpha^*\psi(j)$. However $\alpha_*\psi(i) = \alpha^*\psi(j)$ if and only if $M(\alpha)\psi(i) = \psi(j)A(\alpha)$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
A(i) & \xrightarrow{\psi(i)} & M(i) \\
\downarrow A(\alpha) & & \downarrow M(\alpha) \\
A(j) & \xrightarrow{\psi(j)} & M(j)
\end{array}$$

Hence $\psi \in \mathfrak{Det}(A, M)$. This tells us that the sequence above is exact. Hence $H^0(I, A, M) = \mathfrak{Det}(A, M)$.

To prove the second statement, let us consider

$$\begin{aligned} D(\alpha : i \rightarrow j) &:= \mathfrak{Det}(A(i), \alpha^* M(j)) \\ &= Der\left(\prod_{\beta: y \rightarrow i} GZ(y), \alpha^* M(j)\right) \\ &= \prod_{\beta: y \rightarrow i} Der(GZ(y), \beta^* \alpha^* M(j)), \text{ by lemma 2.3} \end{aligned}$$

Define D_y for a fixed object $y \in I$ to be a natural system on I (using theorem 4.1) given by:

$$D_y(\alpha : i \rightarrow j) = \prod_{\beta: y \rightarrow i} Der(GZ(y), \beta^* \alpha^* M(j))$$

So one has that:

$$D(i \rightarrow j) = \prod_y D_y(i \rightarrow j)$$

Hence,

$$H_{BW}^*(I, D) = \prod_{y \in I} H_{BW}^*(I, D_y)$$

Now consider the cochain complex $C_{BW}^*(I, D_y)$:

$$\begin{aligned} C_{BW}^*(I, D_y) &= \prod_i D_y(i \rightarrow i) \longrightarrow \prod_{\alpha: i \rightarrow j} D_y(i \rightarrow j) \longrightarrow \dots \\ &= \prod_i \prod_{\beta: y \rightarrow i} Der(GZ(y), \beta^* M(i)) \longrightarrow \prod_{\alpha: i \rightarrow j} \prod_{\beta: y \rightarrow i} Der(GZ(y), \beta^* \alpha^* M(j)) \longrightarrow \dots \end{aligned}$$

$UZ(y)$ forms a basis of the free object $GZ(y)$, applying lemma 3.3, one can rewrite the cochain complex as:

$$C_{BW}^*(I, D_y) = \prod_{y \rightarrow i} \prod_{m \in UZ(y)} A_{\beta j(m)} \longrightarrow \prod_{\alpha: i \rightarrow j} \prod_{\beta: y \rightarrow i} \prod_{m \in UZ(y)} A_{\alpha \beta j(m)} \longrightarrow \dots$$

where $A_{\beta j(m)}$ = preimage of $\beta \gamma(m)$ in the projection $M(j) \rightarrow GZ(j)$. This allows us to rewrite the cochain complex as

$$C_{BW}^*(I, D_y) = \prod_{m \in UZ(y)} C_{BW}^*(y/I, F_m)$$

where $F_m : y/I \rightarrow Ab$ is a functor defined by $F_m(\beta : y \rightarrow i) = A_{\beta j(m)}$

Since the category y/\mathfrak{C} contains an initial object $(id_y : y \rightarrow y)$, so by lemma 2.1 the cohomology vanishes in positive dimensions. \square

Theorem 4.4 $H_{G_I}^*(A, M) = H^*(I, A, M)$

Proof:. We are going to show that:

$$H^*(C^\bullet(I, A, M)) \simeq H^*(\text{Tot}(C^\bullet(I, G_I(A), M))) \simeq H^*(C_{G_I}^\bullet(A, M))$$

Start by considering $C^\bullet(I, G_I^p(A), M)$. Since $G_I^p(A)$ is \mathbb{G}_I -projective, it follows that

$$H^n(\text{Tot}(C^\bullet(I, G_I(A), M))) = \begin{cases} \mathfrak{Der}(G_I^p(A), M), & n = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Hence by lemma 4.3 $H^*(C^\bullet(I, A, M)) \simeq H^*(\text{Total}(C^\bullet(I, G_I(A), M)))$.

Now let us consider $C^{p,q}(I, G_I(A), M) = \prod_{x_0 \rightarrow \dots \rightarrow x_p} \mathfrak{Der}(G^{q+1}(G_I^*(A)(x_0)), M(x_p))$. One has that $\mathbb{G}_I^*(A) \rightarrow A$, which is an augmented simplicial object and one can apply the functor U_I to get: $U_I G_I^*(A) \rightarrow U_I(A)$ which is contractible in (\mathfrak{Sets}^{I_0}) . Then one can apply the functor F_I to get $G_I \mathbb{G}_I^*(A) \rightarrow G_I(A)$ which is contractible in $\mathfrak{Alg}(T)$. Hence $G_I^{q+1} \mathbb{G}_I^*(A) \rightarrow G_I^{q+1}(A)$ is contractible in $\mathfrak{Alg}(T)$. Applying the functor $\mathfrak{Der}(-, M)$, one gets a contractible cosimplicial abelian group. Hence $H^*(\text{Tot}(C^\bullet(I, G_I(A), M))) \simeq H^*(C_{G_I}^\bullet(A, M))$. \square

Now one has both a global cohomology, $H_{\mathbb{G}_I}^*(A, M)$, and a local cohomology, $H^*(A(i), M(i))$. One can ask how these two are related; the answer is given by the local to global spectral sequence:

$$E_2^{pq} = H_{BW}^p(I, \mathcal{H}^q(A, M)) \Rightarrow H_{\mathbb{G}_I}^{p+q}(A, M).$$

where $\mathcal{H}^q(A, M)$ is a natural system on I whose value on $(\alpha : i \rightarrow j)$ is given by $H^q(A(i), M(j))$.

5 Applications

5.1 Ψ -rings

As an example of the general theory, one can consider Ψ -rings. A Ψ -ring is a commutative ring R with identity 1, with a sequence of ring homomorphisms $\Psi^n : R \rightarrow R$, $n \geq 1$ satisfying $\forall x \in R$, and integers $n, m \geq 1$.

1. $\Psi^1(x) = x$
2. $\Psi^n(\Psi^m(x)) = \Psi^{nm}(x)$

So to know Ψ -rings it is sufficient to know Ψ^p , for p prime such that for all primes p, q , $\Psi^p \Psi^q = \Psi^q \Psi^p$.

If R is a Ψ -ring, M is a Ψ -module if M is an R -module together with a sequence of homomorphisms $\Psi^n : M \rightarrow M$ such that for all $m \in M$, $r \in R$, $l, n \geq 1$:

1. $\Psi^1(m) = m$
2. $\Psi^n(rm) = \Psi^n(r)\Psi^n(m) = \Psi^n(m)\Psi^n(r)$

$$3. \Psi^n(\Psi^l(m)) = \Psi^{nl}(m)$$

We let $R\text{-mod}_\Psi$ denote the category of all Ψ -modules over R .

Let R be a Ψ -ring, $M \in R\text{-mod}_\Psi$ then the semidirect product of the underlying ring and module, $R \rtimes M$, together with maps: $\Psi^i : R \rtimes M \rightarrow R \rtimes M$ for $i \geq 1$ given by:

$$\Psi^i(r, m) = (\Psi^i(r), \Psi^i(m))$$

is a Ψ -ring. We call this the *semi-direct product* of R and M , denoted by $R \rtimes_\Psi M$

We define a Ψ -derivation is a Ψ -module homomorphism $d : R \rightarrow M$ such that $\forall r, r' \in R, \forall n \geq 1$:

$$d(rr') = rd(r') + d(r)r'$$

$$\Psi^n(d(r)) = d(\Psi^n(r))$$

We let $Der_\Psi(R, M)$ denote the set of all Ψ -derivations $d : R \rightarrow M$. One would expect the following theorem:

Theorem 5.1 *There is a one-to-one correspondence between the sections of $R \rtimes_\Psi M \xrightarrow{\pi} R$ and the Ψ -derivations $d : R \rightarrow M$*

Proof of theorem. Assume we have a section of π , then:

$$R \rtimes_\Psi M \begin{matrix} \xrightarrow{\pi} \\ \xleftarrow{\sigma} \end{matrix} R$$

$\sigma\pi = id_R$, so $\sigma(x) = (x, d(x))$ for some $d : R \rightarrow M$.

$$d(x+y) = d(x) + d(y), \quad d(xy) = d(x)y + xd(y)$$

follow from σ being a homomorphism of Ψ -rings. σ preserves the Ψ -ring structure, meaning that $\Psi^i\sigma(x) = \sigma\Psi^i(x)$.

$$\Psi^i\sigma(x) = \Psi^i(x, d(x)) = (\Psi^i(x), \Psi^i(d(x)))$$

$$\sigma\Psi^i(x) = (\Psi^i(x), d(\Psi^i(x)))$$

Hence $\Psi^i\sigma(x) = \sigma\Psi^i(x)$ if and only if $\Psi^i d(x) = d\Psi^i(x)$. This tells us that if σ is a section of π , then we have a Ψ -derivation d .

Conversely, if we have a Ψ -derivation $d : R \rightarrow M$, then $\sigma(x) = (x, d(x))$ is a section of π . \square

We now construct the free Ψ -ring on one generator a . Let A be the free commutative ring generated by a_0, a_1, a_2, \dots . Since there are countably infinitely many primes, it is possible to label them with the natural numbers. Set $a_0 = a$, and $a_i = \Psi^p(a)$, where p is the i^{th} prime, for $i \in \mathbb{N}$. Then A is a Ψ -ring.

More generally, we can construct a free Ψ -ring on generators a, b, c, \dots, n . We let R be the free commutative ring generated by $a_0, a_1, \dots, b_0, b_1, \dots, c_0, c_1, \dots, n_0, n_1, \dots$. Set $a_0 = a, b_0 = b, c_0 = c, \dots, n_0 = n$, and $a_i = \Psi^p(a), b_i = \Psi^p(b), c_i = \Psi^p(c), \dots, n_i = \Psi^p(n)$ where p is the i^{th} prime, for $i \geq 1$. Then R is a Ψ -ring.

There is a forgetful functor $U : \Psi\text{-rings} \rightarrow \mathbf{Sets}$ from the category of Ψ -rings to the category of sets. This has the left adjoint $F : \mathbf{Sets} \rightarrow \Psi\text{-rings}$, where $F(S)$ is the free Ψ -ring generated by $S \in \mathbf{Sets}$. Hence there is an adjoint pair of functors:

$$\mathbf{Sets} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \Psi\text{-rings}$$

where U is the forgetful functor, and F is the free functor. The adjoint pair of functors yields a comonad $\mathbb{G} = FU : \Psi\text{-rings} \rightarrow \Psi\text{-rings}$ which is monadic.

Let \mathbb{N}^{mult} denote the multiplicative monoid of the natural numbers, and let I denote the category with one object associated to \mathbb{N}^{mult} . Then one can consider Ψ -rings as diagrams of algebras being functors from I to the category of commutative rings, $\mathbf{Com.rings}$. So Ψ -rings are diagrams of algebras with \mathbb{N}^{mult} acting on R a commutative ring with identity. Hence we can use the theory which we developed in the previous section.

It is well known that there is an adjoint pair of functors:

$$\mathbf{Sets} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{U} \end{array} \mathbf{Com.rings}$$

This gives rise to a comonad $\mathbb{G} = FU : \mathbf{Com.rings} \rightarrow \mathbf{Com.rings}$ which is monadic and the cohomology with respect to this monad is known to be André-Quillen cohomology. Now we can define a new comonad $G_I(A)(i) = \coprod_{x \rightarrow i} G(A(x))$ on $\mathbf{Com.rings}^I = \Psi\text{-rings}$. Using the bicomplex $C^{*,*}(I, A, M)$ described in the previous section, we can define cohomology of Ψ -rings. If $R : \mathbb{N}^{mult} \rightarrow \mathbf{Com.rings}$ is a Ψ -ring and M is an R -module, then for any $n \geq 0$, there is a natural system on \mathbb{N}^{mult} as follows:

$$D_f := H_{AQ}^n(R, M^f)$$

where M^f is an R -module with M as an abelian group with the following action of R :

$$(r, a) \mapsto \Psi^n(r)\Psi^n(a), \text{ for } r \in R, a \in M$$

For $u \in \mathbb{N}^{mult}$, we have $u_* : D_f \rightarrow D_{uf}$ which is induced by $\Psi^u : M^f \rightarrow M^{uf}$. For $v \in \mathbb{N}^{mult}$, we have $v^* : D_f \rightarrow D_{fv}$ which is induced by $\Psi^v : R \rightarrow R$.

There exists a spectral sequence:

$$E_2^{p,q} = H_{BW}^p(\mathbb{N}^{mult}, H_{AQ}^q(A, M)) \Rightarrow H_{\Psi}^{p+q}(A, M).$$

where $H_{\Psi}^*(A, M)$ denotes the cohomology of Ψ -rings as it is defined via comonads.

5.2 Π -algebras

A Π -algebra is a graded group equipped with the action of primary homotopy operations modeled on the homotopy groups of a space. Dwyer and Kan [9]

defined the Quillen cohomology of Π -algebras, which we denote by $H_{DK}^*(A, M)$. Blanc, Johnson, and Turner [6], defined the Quillen cohomology of diagrams of Π -algebras, which we denote by $H_{BJT}^*(A, M)$. However, it is known that Quillen's and Beck's approaches yield the same cohomology.

An application of our main result is that there exists a spectral sequence:

$$E_2^{p,q} = H_{BW}^p(I, \mathcal{H}_{DK}^q(A, M)) \Rightarrow H_{BJT}^{p+q}(A, M).$$

where $\mathcal{H}_{BJT}^*(A, M)$ is the natural system on I whose value on $\alpha : i \rightarrow j$ is given by $H_{DK}^*(A(i), \alpha^* M(j))$.

If we let I be the small category with two distinct objects $0, 1$ and one non-trivial map $0 \rightarrow 1$, then our spectral sequence yields corollary 4.27 in [6].

5.3 Diagrams of groups

In the paper by Cegarra [7], the cohomology of diagrams of groups is described, which we denote by $H_C^*(G, A)$. There is also described the following spectral sequence:

Let I be a small category. If $G : I \rightarrow \mathbf{Gp}$ is an I -group and A is a G -module, then for any $n \geq 0$, there is a natural system on I as follows:

$$\mathcal{H}^n(G, A) : \mathcal{FI} \rightarrow \mathbf{Ab}, \quad u \xrightarrow{\sigma} v \mapsto \begin{cases} H^n(G(u), A(v)) & \text{if } n \geq 2; \\ \text{Der}(G(u), A(v)) & \text{if } n = 1; \\ 0, & \text{if } n = 0. \end{cases}$$

Then there is a natural spectral sequence:

$$E_2^{p,q} = H_{BW}^p(I, \mathcal{H}^{q+1}(G, A)) \Rightarrow H_C^{p+q+1}(G, A)$$

where $\mathcal{H}^{q+1}(G, A)$ is the natural system on I as described above.

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